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Dynamo action associated with random inertial waves in a rotating conducting fluid

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It is shown that a random superposition of inertial waves in a rotating conducting fluid can act as a dynamo, i.e. can systematically transfer energy to a magnetic field which has no source other than electric currents within the fluid. Dynamo action occurs provided the statistical properties of the velocity field lack reflexional symmetry, and this occurs when conditions are such that there is a net energy flux (positive or negative) in the direction of the rotation vector $\mathbf{\Omega}$.

If the magnetic field grows from an infinitesimal level, then the mode of maximum growth rate dominates before the back-reaction associated with the Lorentz force becomes significant. This mode is first determined, and then the back-reaction associated with it alone is analysed. It is shown that the magnetic energy grows exponentially during the stage when the Lorentz forces are negligible, then reaches a maximum depending on the values of the parameters

$$R_m = u_0 l/\lambda, \quad Q = \Omega l^2/\lambda,$$

(u_0 = initial r.m.s. velocity, l = length scale characteristic of the velocity field, λ = magnetic diffusivity) and ultimately decays as t^{-1} (equation (5.15)). This decay is coupled with a decay of the velocity field due to ohmic dissipation, and it occurs because there is no external source of energy for the fluid motion.

1. Introduction

In a previous paper (Moffatt 1970, hereafter referred to as I), the effect of turbulence on a weak magnetic field in an electrically conducting fluid was considered; and it was shown that when the magnetic Reynolds number $R_m = u_0 l/\lambda$ is small (u_0 = r.m.s. velocity, l = length scale of energy containing eddies, λ = magnetic diffusivity), exponentially growing magnetic modes are possible, provided the statistical properties of the turbulence lack reflexional symmetry. The growing magnetic field has no source other than electric currents flowing within the fluid itself. It has a length scale L large compared with l ($L = O(R_m^{-2})\, l$) and a time scale t_1 (doubling time) large compared with the time scale $t_0 = l/u_0$ characteristic of the turbulence ($t_1 = O(R_m^{-3})\, t_0$). The exponential growth continues only for so long as the back-reaction of the Lorentz force on the fluid can be neglected; this (dynamic) aspect of the problem was not investigated in I.

In the present paper the investigation is extended and specialized to a situation that may be of particular relevance and importance in geophysical and astrophysical contexts. We suppose that the 'turbulent' velocity field $\mathbf{u}(\mathbf{x}, t)$ consists of a random superposition of inertial waves in a fluid rotating with uniform angular velocity $\mathbf{\Omega}$. It will be supposed that the Rossby number is small, i.e.

$$R_0 = u_0/\Omega l \ll 1, \tag{1.1}$$

so that inertial interactions between waves of different wave-numbers may be neglected. Further, viscosity will be neglected, on the grounds that viscous dissipation is likely to be dominated by ohmic dissipation in situations of practical interest. It will first be shown that such a motion is capable of amplifying a magnetic field $\mathbf{B}(\mathbf{x}, t)$ on scales L, T large compares with l and t, through much the same mechanism as described in I. The nature of the amplification processes is largely governed by the values of the parameter $Q = \Omega l^2/\lambda = R_m R_0^{-1}$. If

$$Q \ll 1, \tag{1.2}$$

(so that, by (1.1)
$$R_m \ll 1 \tag{1.3}$$

also), then the process is identical with that of I; but if

$$Q \gg 1, \tag{1.4}$$

(the value of R_m being unrestricted) there are some important differences; e.g. the growth of the magnetic field shows strong directional preferences, even if the amplitudes of the inertial waves are isotropically distributed.

As the field grows in strength, it begins to react back upon the constituent inertial waves of the velocity field in a manner that can be explicitly taken into account. The most significant effect is that each inertial wave, through its coupling with the magnetic field (which may be regarded as uniform and steady over scales characteristic of the velocity field), loses energy to the ohmic sink. In the absence of any body force distribution, the velocity field therefore decays as the magnetic field grows. Ultimately the magnetic field must likewise decay to zero. We are primarily interested in the maximum level attained by the field before this ultimate decay sets in. In the absence of any input of energy, the magnetic field clearly cannot acquire more energy than is released from the velocity field; for a random magnetic field, this implies

$$(\mu\rho)^{-1}\langle \mathbf{B}^2\rangle < u_0^2 - \overline{u^2}, \tag{1.5}$$

where u_0 is now the 'initial' ($t \to -\infty$) r.m.s. velocity, μ and ρ are respectively the magnetic permeability and the density of the fluid, and the angular brackets $\langle \ldots \rangle$ are used to mean an average over scales large compared with L. Actually, it will turn out that in none of the circumstances considered can the magnetic field acquire more than a small fraction of the initial kinetic energy associated with the wave motion.

As in I, it will be convenient to write

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t), \tag{1.6}$$

where $\mathbf{B}_0 = \overline{\mathbf{B}}$ varies only on scales L, T, and $\mathbf{b}$ is the small fluctuation field induced by the motion $\mathbf{u}$ across $\mathbf{B}_0$; it will emerge in §2 that the condition (1.1) ensures that

$$|\mathbf{b}| \ll |\mathbf{B}_0|. \tag{1.7}$$

It will further be convenient to use the equivalent Alfvén velocities

$$\mathbf{h}_0 = (\mu\rho)^{-\frac{1}{2}}\mathbf{B}_0, \quad \mathbf{h} = (\mu\rho)^{-\frac{1}{2}}\mathbf{b}, \tag{1.8}$$

as a measure of the magnetic field. As in I, the large scale field $\mathbf{h}_0$ evolves according to the equation

$$\frac{\partial \mathbf{h}_0}{\partial t} = \nabla \wedge \overline{\mathbf{u} \wedge \mathbf{h}} + \lambda\nabla^2\mathbf{h}_0, \tag{1.9}$$

and the main objective is to find an expression for $\overline{\mathbf{u} \wedge \mathbf{h}}$ in terms of the initial properties of the velocity field, and of $\mathbf{h}_0$ and t; and then to solve (1.9). To find $\overline{\mathbf{u} \wedge \mathbf{h}}$ we need to investigate some of the detailed properties of inertial waves in an (apparently) uniform steady magnetic field $\mathbf{h}_0$, and this is done in the following section.

The possibility of dynamo action due to a mechanism of the type analysed in this paper was first explored by Steenbeck, Krause & Rädler (1966). The idea was further developed in a series of papers by the same authors, full references to which are given in I. In none of these papers however was the back-reaction of the growing magnetic field taken into account; and an understanding of this process is the chief objective of the present investigation.

2. Inertial waves in the presence of a uniform steady magnetic field

The linearized equations governing inertial waves modified by a magnetic field $\mathbf{h}_0$ are (Lehnert 1954; Chandrasekhar 1961, §50)

$$\partial\mathbf{u}/\partial t + 2\mathbf{\Omega} \wedge \mathbf{u} = -\nabla\chi + \mathbf{h}_0 . \nabla\mathbf{h}, \tag{2.1}$$

$$\partial\mathbf{h}/\partial t = \mathbf{h}_0 . \nabla\mathbf{u} + \lambda\nabla^2\mathbf{h}, \tag{2.2}$$

$$\nabla . \mathbf{u} = \nabla . \mathbf{h} = 0. \tag{2.3}$$

The linearization is valid provided $|\mathbf{u}| \ll \Omega l$ and $|\mathbf{h}| \ll |\mathbf{h}_0|$. $\chi(\mathbf{x}, t)$ is a reduced pressure distribution, modified by the centrifugal force, and the irrotational part of the Lorentz force:

$$\chi = p/\rho + (2\mu)^{-1}\mathbf{B}^2 - (\mathbf{\Omega} \wedge \mathbf{x})^2. \tag{2.4}$$

Equations (2.1)–(2.3) admit plane wave solutions of the form

$$(\mathbf{u}, \mathbf{h}, \chi) = (\hat{\mathbf{u}}, \hat{\mathbf{h}}, \hat{\chi}) \exp\{i(\mathbf{k} . \mathbf{x} - \omega t)\}, \tag{2.5}$$

where

$$-i\omega\hat{\mathbf{u}} + 2\mathbf{\Omega} \wedge \hat{\mathbf{u}} = -i\mathbf{k}\hat{\chi} + i(\mathbf{h}_0 . \mathbf{k})\hat{\mathbf{h}}, \tag{2.6}$$

$$-i\omega\hat{\mathbf{h}} = i(\mathbf{h}_0 . \mathbf{k})\hat{\mathbf{u}} - \lambda k^2\hat{\mathbf{h}}, \tag{2.7}$$

$$\mathbf{k} . \hat{\mathbf{u}} = \mathbf{k} . \hat{\mathbf{h}} = 0. \tag{2.8}$$

Such waves degenerate to pure inertial waves in the limit $\mathbf{h}_0 \to 0$, and to damped Alfvén waves in the limit $\mathbf{\Omega} \to 0$. Equation (2.7) gives the important relation

$$\hat{\mathbf{h}} = -\frac{\mathbf{h}_0 . \mathbf{k}}{\omega + i\lambda k^2}\hat{\mathbf{u}}, \tag{2.9}$$

between $\hat{\mathbf{h}}$ and $\hat{\mathbf{u}}$; and (2.6) then gives

$$i\sigma\hat{\mathbf{u}} + 2\mathbf{\Omega} \wedge \hat{\mathbf{u}} = -i\mathbf{k}\hat{\chi}, \tag{2.10}$$

where

$$\sigma = -\omega + (\mathbf{h}_0 . \mathbf{k})^2(\omega + i\lambda k^2)^{-1}. \tag{2.11}$$

From (2.10) using $\mathbf{k}.\hat{\mathbf{u}} = 0$, we have

$$i\sigma\mathbf{k}\wedge\hat{\mathbf{u}} - 2(\mathbf{k}.\boldsymbol{\Omega})\,\hat{\mathbf{u}} = 0, \tag{2.12}$$

and, crossing again with $\mathbf{k}$,

$$i\sigma k^2\hat{\mathbf{u}} + 2(\mathbf{k}.\boldsymbol{\Omega})\,\mathbf{k}\wedge\hat{\mathbf{u}} = 0, \tag{2.13}$$

so that, eliminating $\hat{\mathbf{u}}$,

$$\sigma = \pm\, 2(\mathbf{k}.\boldsymbol{\Omega})/k, \tag{2.14}$$

and, correspondingly,

$$\hat{\boldsymbol{\omega}} = i\mathbf{k}\wedge\hat{\mathbf{u}} = \pm\, k\hat{\mathbf{u}}, \tag{2.15}$$

where $\hat{\boldsymbol{\omega}}$ is the vorticity Fourier component corresponding to $\hat{\mathbf{u}}$.

The situation when $\mathbf{h}_0 \to 0$

When $\mathbf{h}_0 = 0$, $\sigma = -\omega$, so that from (2.14),

$$\omega = \mp\, 2(\mathbf{k}.\boldsymbol{\Omega})/k = \mp\, 2\Omega\cos\theta, \tag{2.16}$$

where θ is the angle between $\boldsymbol{\Omega}$ and $\mathbf{k}$. This is the well-known dispersion relation for pure inertial waves (Greenspan 1968). The phase velocity is $\omega\mathbf{k}/k^2$, and the group velocity is

$$\mathbf{c}_g = \nabla_{\mathbf{k}}\omega = \mp\frac{2}{k^3}(\boldsymbol{\Omega}k^2 - (\boldsymbol{\Omega}.\mathbf{k})\,k) = \mp\, 2\boldsymbol{\Omega}_\perp/k, \tag{2.17}$$

where $\boldsymbol{\Omega}_\perp$ is the component of $\boldsymbol{\Omega}$ perpendicular to $\mathbf{k}$ (figure 1(*a*)). Hence, if $\mathbf{k}.\boldsymbol{\Omega} > 0$, the upper and lower signs correspond to propagation 'downwards' and 'upwards' relative to the direction of $\boldsymbol{\Omega}$.

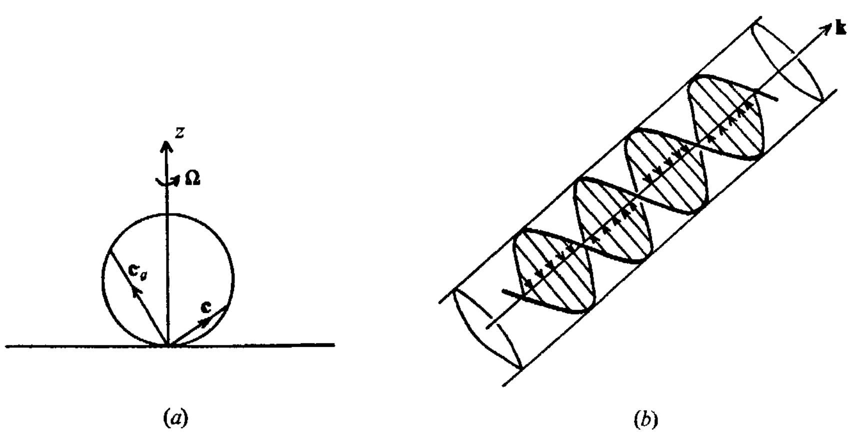

FIGURE 1. (*a*) The phase velocity $\mathbf{c} = 2(\mathbf{k}.\boldsymbol{\Omega})\,\mathbf{k}/k^3$, and the group velocity $\mathbf{c}_g = 2\mathbf{k}\wedge(\boldsymbol{\Omega}\wedge\mathbf{k})/k^3$ for inertial waves with energy flux in the positive z-direction. (*b*) The velocity field in a single inertial wave (with or without magnetic field).

A random superposition of such waves will exhibit a lack of reflexional symmetry only if there is a net energy flux upwards or downwards, i.e. only if there are more of the upward propagating waves than the downward (or vice-versa). Such a situation could arise, for example, if the waves were generated in the half space $z > 0$ by random mechanical excitation on the plane $z = 0$; in this case, only the upward propagating waves would be present. We shall assume that only such waves are present in what follows.

The velocity field in a typical upward propagating wave is indicated in figure 1 (*b*). The streamlines are straight, and their direction rotates clockwise in the direction of the wave-vector $\mathbf{k}$. The particle paths are circles in planes perpendicular to $\mathbf{k}$.

A measure of the lack of reflexional symmetry in a single wave is provided by the *helicity* $\overline{\mathbf{u}.\boldsymbol{\omega}} = \overline{\mathbf{u}.(\nabla\wedge\mathbf{u})}$. From (2.15),

$$\overline{\mathbf{u}.\boldsymbol{\omega}} = \tfrac{1}{2}\mathscr{R}\hat{\mathbf{u}}.\hat{\boldsymbol{\omega}} = \pm\tfrac{1}{2}k|\hat{\mathbf{u}}|^2, \tag{2.18}$$

so that the helicity of upward propagating waves is negative. A random superposition of upward propagating waves gives a velocity field.

$$\mathbf{u}(\mathbf{x},t) = \mathscr{R}\int \hat{\mathbf{u}}(\mathbf{k})\exp\{i(\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t)\}\,d^3\mathbf{k}, \tag{2.19}$$

wherein we may suppose $\mathbf{k}.\boldsymbol{\Omega} > 0$, and where, in the limit $\mathbf{h}_0 \to 0$,

$$\omega(\mathbf{k}) = 2(\mathbf{k}.\boldsymbol{\Omega})/k = 2\Omega\cos\theta. \tag{2.20}$$

The spectrum tensor of this velocity field is

$$\Phi_{ij}(\mathbf{k}) = \lim_{d^3\mathbf{k}\to 0} \overline{\hat{u}_i^*(\mathbf{k})\,\hat{u}_j(\mathbf{k})}\,d^3\mathbf{k}. \tag{2.21}$$†

If the amplitudes of the waves are isotropically distributed, this must take the form

$$\Phi_{ij}(\mathbf{k}) = \frac{E(k)}{2\pi k^4}(k^2\delta_{ij}-k_i\,k_j)+\frac{iF(k)}{4\pi k^4}\epsilon_{ijk}k_k, \tag{2.22}$$

where
$$E(k) = \pi k^2\Phi_{ii}(\mathbf{k}) = \pi k^2 \lim_{d^3\mathbf{k}\to 0}\overline{|\hat{\mathbf{u}}(\mathbf{k})|^2}\,d^3\mathbf{k} \tag{2.23}$$
is the energy spectrum function, and

$$F(k) = -2\pi k^2\epsilon_{ijk}ik_k\,\Phi_{ij}(\mathbf{k}) = 2\pi k^2 \lim_{d^3\mathbf{k}\to 0}\overline{\hat{u}_i^*\hat{\omega}_i}\,d^3\mathbf{k} \tag{2.24}$$

is the *helicity spectrum function*. The assumption that there are only upward propagating waves means that

$$i\mathbf{k}\wedge\hat{\mathbf{u}} = \hat{\boldsymbol{\omega}}(\mathbf{k}) = -k\hat{\mathbf{u}}(\mathbf{k}), \tag{2.25}$$

so that
$$F(k) = -2kE(k). \tag{2.26}$$‡

The velocity scale u_0 and length scale l characteristic of the wave field may now be *defined* by

$$\tfrac{1}{2}\overline{\mathbf{u}^2} = \int_0^\infty E(k)\,dk = \tfrac{1}{2}u_0^2, \tag{2.27}$$

and
$$\overline{\mathbf{u}.\boldsymbol{\omega}} = \int_0^\infty F(k)\,dk = -2\int_0^\infty kE(k)\,dk = -u_0^2/l. \tag{2.28}$$

† When the overbar appears in such an expression, it must be interpreted as an ensemble average, which is identical with the space average for homogeneous turbulence.

‡ More generally, if a mixture of upward and downward propagating waves were considered, a relation of the form
$$F(k) = 2a(k)\,E(k),$$
where $|a(k)| < 1$, would hold.

If u_0 and l are the *only* scales characterizing the field, then on dimensional grounds

$$E(k) = \tfrac{1}{2}u_0^2 l f(kl), \tag{2.29}$$

and (2.27) and (2.28) then become

$$\int_0^\infty f(\eta)\,d\eta = \int_0^\infty \eta f(\eta)\,d\eta = 1. \tag{2.30}$$

A particular form of $f(\eta)$, satisfying these constraints, that will be used by way of illustration in what follows, is

$$f(\eta) = \delta(\eta - 1), \tag{2.31}\dagger$$

corresponding to a sea of waves, randomly oriented, but all having the same wavelength $2\pi l$.

Dynamic influence of the magnetic field

When $\mathbf{h}_0 \neq 0$, (2.11) and (2.14) (lower sign) give

$$\omega = 2\Omega\cos\theta + \frac{(\mathbf{h}_0.\mathbf{k})^2}{\omega + i\lambda k^2}. \tag{2.32}$$

The condition $u_0 \ll \Omega l$, or equivalently $u_0 k \ll \Omega$ for all k giving significant contributions to the integral (2.27), and the gross energy constraint (from (1.5)),

$$|\mathbf{h}_0| < u_0, \tag{2.33}$$

together imply that

$$|\mathbf{h}_0.\mathbf{k}| \ll \Omega. \tag{2.34}$$

Hence (2.29) implies that, except possibly when $\theta \approx \frac{1}{2}\pi$, the magnetic field causes a small modification in ω:

$$\omega \approx 2\Omega\cos\theta + \frac{(\mathbf{h}_0.\mathbf{k})^2}{2\Omega\cos\theta + i\lambda k^2}, \tag{2.35}$$

and it is evident that this approximation is valid provided

$$|\mathbf{h}_0.\mathbf{k}| \ll (4\Omega^2\cos^2\theta + \lambda^2 k^4)^{\frac{1}{2}}. \tag{2.36}$$

We shall use (2.35) in what follows, and investigate the limits of its validity in §5.

It is evident from (2.35) that when $\mathbf{h}_0.\mathbf{k} \neq 0$, ω becomes complex, implying a damping of the waves; indeed if $\omega = \omega_r + i\omega_i$, then the lowest approximation for ω_r and ω_i is

$$\omega_r \approx 2\Omega\cos\theta, \quad \omega_i \approx \frac{-(\mathbf{h}_0.\mathbf{k})^2\lambda k^2}{4\Omega^2\cos^2\theta + \lambda^2 k^4}. \tag{2.37}$$

Note that waves for which $\mathbf{h}_0.\mathbf{k} = 0$ are not damped; such waves show no tendency to bend the magnetic lines of force, and they therefore cannot feel the effect of ohmic dissipation.

† A more realistic form, satisfying (2.30) and the necessary kinematic constraint near $\eta = 0$, might be

$$f(\eta) = C\eta^4 e^{-5\eta}, \quad C = 5^5/4!,$$

but there seems little point in complicating the analysis in this paper by the use of such a function.

3. The expression for $\overline{\mathbf{u} \wedge \mathbf{h}}$

We are now in a position to calculate

$$\overline{\mathbf{u} \wedge \mathbf{h}} = \tfrac{1}{2}\mathscr{R} \int \overline{\hat{\mathbf{u}} \wedge \hat{\mathbf{h}}^* d^3\mathbf{k}}. \tag{3.1}$$

Using (2.9) we have

$$\hat{\mathbf{u}} \wedge \hat{\mathbf{h}}^* = -\frac{\mathbf{h}_0 . \mathbf{k}}{\omega^* - i\lambda k^2} \hat{\mathbf{u}} \wedge \hat{\mathbf{u}}^*, \tag{3.2}$$

and, using (2.25) and $\mathbf{k}.\hat{\mathbf{u}}(\mathbf{k}) = 0$,

$$\hat{\mathbf{u}} \wedge \hat{\mathbf{u}}^* = \frac{i}{k} \hat{\mathbf{u}} \wedge (\mathbf{k} \wedge \hat{\mathbf{u}}^*) = \frac{i\mathbf{k}}{k} |\hat{\mathbf{u}}|^2 e^{2\omega_i t}. \tag{3.3}$$

Hence

$$\mathscr{R}\overline{\hat{\mathbf{u}} \wedge \hat{\mathbf{h}}^*} = \frac{\lambda k (\mathbf{h}_0 . \mathbf{k})\, \mathbf{k} E(k)\, e^{2\omega_i t}}{\pi k^2 |\omega + i\lambda k^2|^2}. \tag{3.4}$$

Under the condition (2.36) for all relevant $\mathbf{k}$, we therefore have

$$(\overline{\mathbf{u} \wedge \mathbf{h}})_i = \lambda^{-1} A_{ij} h_{0j}, \tag{3.5}$$

where

$$A_{ij} = \frac{1}{2\pi} \int \frac{E(k)\, k_i k_j \lambda^2}{k(\lambda^2 k^4 + 4\Omega^2 \cos^2\theta)} \exp\left\{\frac{-2\lambda k^2 (\mathbf{h}_0 . \mathbf{k})^2 t}{\lambda^2 k^4 + 4\Omega^2 \cos^2\theta}\right\} d^3\mathbf{k}. \tag{3.6}$$

If the spectrum function $E(k)$ has the form (2.29), with $f(\eta)$ given by (2.31), then putting

$$\mathbf{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = k\hat{\mathbf{k}}, \tag{3.7}$$

(3.6) becomes

$$A_{ij} = \frac{u_0^2 l}{4\pi} \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} \frac{\hat{k}_i \hat{k}_j}{1 + 4Q^2 \cos^2\theta} \exp\left\{\frac{-2\lambda^{-1} (\mathbf{h}_0 . \hat{\mathbf{k}})^2 t}{1 + 4Q^2 \cos^2\theta}\right\} \sin\theta\, d\theta\, d\phi. \tag{3.8}$$

Evidently, A_{ij} is a real symmetric tensor, depending on the parameters Q and $h_0^2 t/\lambda$, and on the orientation of the vector $\mathbf{h}_0$ relative to $\mathbf{\Omega}$. We shall first consider (in §4) the form of A_{ij} when $\mathbf{h}_0$ is so weak that the exponential factor in the integrand is effectively unity; and we shall show that in these circumstances dynamo action occurs for all values of the parameter Q. This means that $\mathbf{h}_0$ grows exponentially so that at some stage the exponential factor in the integrand becomes important in restricting the growth of $\mathbf{h}_0$. This effect will be examined in detail in §5.

4. Dynamo action during the stage of negligible Lorentz forces

Neglect of the Lorentz force is equivalent to taking the limit $\mathbf{h}_0 \to 0$ in (3.8), i.e. to omitting the exponential factor. The ϕ-integration is then trivial and we have

$$A_{ij} = u_0^2 l [\alpha_0(Q)\, \delta_{ij} + (\gamma_0(Q) - \alpha_0(Q))\, \Omega_i \Omega_j / \Omega^2], \tag{4.1}$$

where

$$\alpha_0(Q) = \frac{1}{4} \int_0^{\frac{1}{2}\pi} \frac{\sin^3\theta\, d\theta}{1 + 4Q^2\cos^2\theta} = \frac{1}{8Q}\left[\left(1 + \frac{1}{4Q^2}\right)\tan^{-1} 2Q - \frac{1}{2Q}\right], \tag{4.2}$$

$$\gamma_0(Q) = \frac{1}{2} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta \sin\theta\, d\theta}{1 + 4Q^2\cos^2\theta} = \frac{1}{8Q^2}\left[1 - \frac{1}{2Q}\tan^{-1} 2Q\right]. \tag{4.3}$$

These functions are sketched in figure 2. Note that

$$\alpha_0(0) = \gamma_0(0) = \tfrac{1}{6}, \tag{4.4}$$

and that, as $Q \to \infty$, $\quad \alpha_0(Q) \sim \pi/2Q, \quad \gamma_0(Q) \sim 1/8Q^2. \qquad (4.5)$

Also, for all $Q > 0$, $\quad \alpha_0(Q) > \gamma_0(Q). \qquad (4.6)$

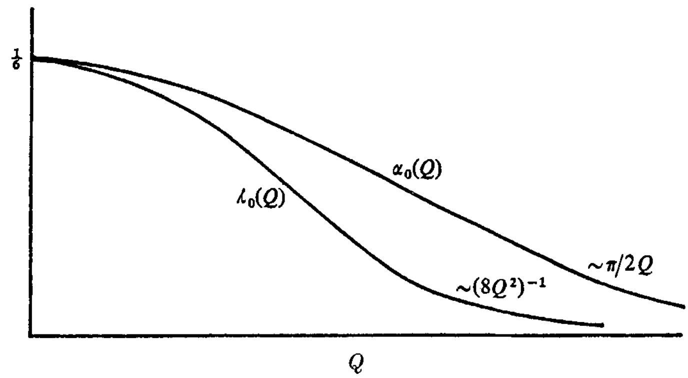

FIGURE 2. The functions $\alpha_0(Q)$ and $\gamma_0(Q)$ defined in (4.2), (4.3).

In the limit $Q \to 0$, A_{ij} becomes isotropic:

$$A_{ij} = \tfrac{1}{6}u_0^2 l\delta_{ij}, \tag{4.7}$$

and this limit corresponds to the situation considered in I. For $Q \neq 0$ however, A_{ij} is anisotropic (though still axisymmetric about the direction of $\mathbf{\Omega}$) even if the amplitudes of the inertial waves are isotropically distributed; this arises of course as a direct result of the anisotropy of the dispersion relation (2.16).

From (3.5) and (4.1) we now have

$$\overline{\mathbf{u} \wedge \mathbf{h}} = \lambda^{-1}[\alpha_1 \mathbf{h}_0 + (\gamma_1 - \alpha_1)(\mathbf{\Omega} \cdot \mathbf{h}_0)\mathbf{\Omega}/\Omega^2], \tag{4.8}$$

where $\alpha_1 = u_0^2 l\alpha_0$, $\gamma_1 = u_0^2 l\gamma_0$, and so equation (1.9) becomes

$$\frac{\partial \mathbf{h}_0}{\partial t} = \frac{\alpha_1}{\lambda}\nabla \wedge \mathbf{h}_0 - \frac{(\gamma_1 - \alpha_1)}{\lambda}(\mathbf{\Omega} \wedge \nabla)\frac{(\mathbf{\Omega} \cdot \mathbf{h}_0)}{\Omega^2} + \lambda\nabla^2 h_0, \tag{4.9}$$

an equation linear in $\mathbf{h}_0$, with constant coefficients. This equation admits 'wave-type' solutions of the form

$$\mathbf{h}_0 = \mathscr{R}(\hat{\mathbf{h}}_0 e^{i\mathbf{K}\cdot\mathbf{x}} e^{mt}), \quad \mathbf{K} \cdot \hat{\mathbf{h}}_0 = 0, \tag{4.10}$$

where (cf. I, § 6)

$$m = -\lambda K^2 \pm \lambda^{-1}\{\alpha_1\gamma_1(K_1^2 + K_2^2) + \alpha_1^2 K_3^2\}^{\frac{1}{2}}. \tag{4.11}$$

The upper sign corresponds to an exponentially growing magnetic mode whenever

$$\alpha_1\gamma_1(K_1^2 + K_2^2) + \alpha_1^2 K_3^2 > \lambda^4 K^4, \tag{4.12}$$

and there are certainly wave-vectors $\mathbf{K}$ for which this inequality holds (figure 3(*a*)).

It may be noticed in passing that the same separation of wave-number space into a region of amplification and a region of decay occurs in dynamo models in which the velocity field is a simple periodic function of the space variables (Roberts 1969; Childress 1969).

Of particular interest is the mode having maximum growth rate, since it is the one which will dominate in a magnetic field which has been amplified from an infinitesimal level. Since $\gamma_1 < \alpha_1$, the maximum value of m, for given $|\mathbf{K}|$ (taking the upper sign in (4.11)) occurs for

$$\mathbf{K} = (0, 0, K), \text{ so that } \hat{\mathbf{h}}_0 = (\hat{h}_{01}, \hat{h}_{02}, 0) \tag{4.13}$$

and then

$$m = -\lambda K^2 + \alpha_1 K/\lambda. \tag{4.14}$$

Clearly $m > 0$, indicating dynamo action, if $K < \alpha_1/\lambda^2$. Substitution of (4.13) and (4.14) in (4.9) gives

$$\hat{h}_{02} = i\hat{h}_{01},$$

and so (4.10) becomes

$$\mathbf{h}_0 = h_{00}(\cos K(z-z_0), \ -\sin K(z-z_0), 0)\, e^{mt} \tag{4.15}$$

for some z_0. This is a 'force-free' magnetic field (Roberts 1967) with straight lines of force, whose direction rotates (anticlockwise) with increasing z (figure 3(*b*)).

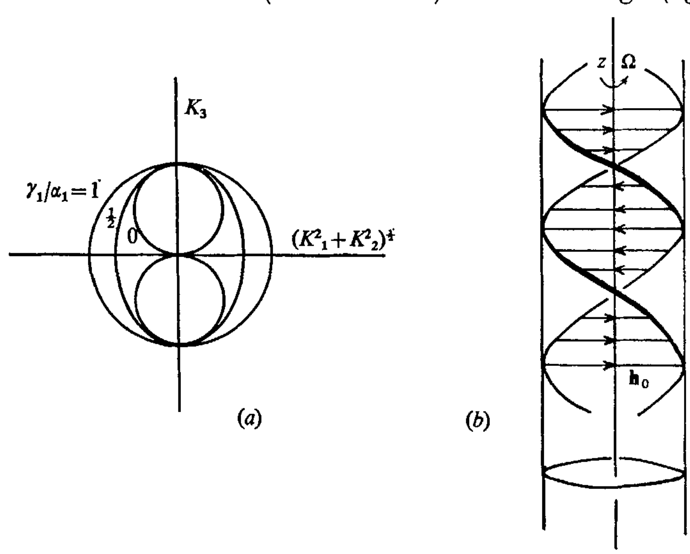

FIGURE 3. (*a*) The surface $\alpha_1\gamma_1(K_1^2+K_2^2)+\alpha_1^2K_3^2 = \lambda^4K^4$ in $\mathbf{K}$-space separating the region of amplification of magnetic modes proportional to $\exp(i\mathbf{K}.\mathbf{x})$ from the region of decay. (*b*) The magnetic mode of maximum growth rate, given by (4.15).

For a magnetic field that has grown from an infinitesimal level, the mode for which m has a maximum value will dominate long before the Lorentz force becomes significant. From (4.14), the maximum value m_0 of m occurs at $K = K_c$ where

$$K_c = \alpha_1/2\lambda^2, \quad m_0 = m(K_c) = \alpha_1^2/4\lambda^3. \tag{4.16}$$

For consistency, these values must satisfy

$$K_c l \ll 1 \quad \text{and} \quad m_0\Omega^{-1} \ll 1; \tag{4.17}$$

the first of these ensures that the length scale of variation of the growing field, $L = O(K_c^{-1})$, is large compared with the length scale of the background velocity field, and the second ensures that the growth rate is slow relative to the time-scale of the velocity field, so that the treatment of §§2 and 3, in which $\mathbf{h}_0$ is treated as locally uniform and steady, is legitimate. With $\alpha_1 = u_0^2 l \alpha_0(Q)$, the conditions (4.17) become

$$R_0^{-2} \gg Q^2 \alpha_0(Q) \quad \text{and} \quad R_0^{-4} \gg Q^3(\alpha_0(Q))^2. \tag{4.18}$$

When $Q = O(1)$ or less, these conditions are both implied by the assumed inequality (1.1). If $Q \gg 1$, however, the first inequality of (4.18), with $\alpha_0(Q) \sim \pi/2Q$, becomes

$$R_0^{-2} \gg Q \gg 1, \tag{4.19}$$

a somewhat stronger condition than (1.1). If $Q \gg R_0^{-2} \gg 1$, there *may* still be dynamo action, but the 'double-length-scale' analysis of this paper would not then appear to be legitimate.

5. The effect of the Lorentz force

The exponential growth of the magnetic field described in §4 cannot continue indefinitely; no matter how weak the initial field may be, at some stage the back-reaction of the Lorentz force on the fluid motion must be taken into account. We shall suppose that there has been sufficient time for the emergence of a dominant mode of the form (4.13), (4.15). The field $\mathbf{h}_0$ defined by (4.15) has the property that h_0^2 is independent of position, and this means that the principal values of the tensor A_{ij} defined by (3.6) or (3.8) remain independent of $\mathbf{x}$ even when the influence of $\mathbf{h}_0$ is included. The principal axes of A_{ij} (with $\mathbf{h}_0 . \mathbf{\Omega} = 0$) are in the directions $\mathbf{\Omega}$, $\mathbf{h}_0$ and $\mathbf{\Omega} \wedge \mathbf{h}_0$, and A_{ij} now has the form

$$(u_0^2 l)^{-1} A_{ij} = \alpha \frac{h_{0i} h_{0j}}{h_0^2} + \beta \left(\delta_{ij} - \frac{h_{0i} h_{0j}}{h_0^2} - \frac{\Omega_i \Omega_j}{\Omega^2} \right) + \gamma \frac{\Omega_i \Omega_j}{\Omega^2}, \tag{5.1}$$

where α, β and γ will now depend on the parameter $S(t) = h_0^2 t / \lambda$ as well as on Q. From (3.5), we then have

$$\overline{\mathbf{u} \wedge \mathbf{h}} = u_0^2 l (\alpha/\lambda)\, \mathbf{h}_0, \tag{5.2}$$

so that it will be sufficient to calculate α.

Choosing axes so that, locally,

$$\mathbf{\Omega} = (0, 0, \Omega), \quad \mathbf{h}_0 = (h_0, 0, 0), \tag{5.3}$$

and with $\mathbf{k}$ given by (3.7), we have

$$\alpha(Q, S) = \frac{A_{11}}{u_0^2 l} = \frac{1}{4\pi} \iint \frac{\sin^3\theta \cos^2\phi}{1 + 4Q^2 \cos^2\theta} \exp\left\{ \frac{-2S \sin^2\theta \cos^2\phi}{1 + 4Q^2 \cos^2\theta} \right\} d\theta\, d\phi. \tag{5.4}$$

The asymptotic forms of this function for $Q \to 0$ and $Q \to \infty$ are obtained in the appendix. Note that

$$\alpha(Q, 0) = \alpha_0(Q). \tag{5.5}$$

Substitution of (5.2) in (1.9) gives

$$\frac{\partial h_0}{\partial t} = \frac{u_0^2 l \alpha}{\lambda} \nabla \wedge \mathbf{h}_0 + \lambda \nabla^2 h_0, \tag{5.6}$$

and we now take account of the dependence of α on h_0^2. The growing mode, selected on the linear analysis of §4, satisfies

$$\nabla \wedge \mathbf{h}_0 = K_c \mathbf{h}_0, \quad \nabla^2 \mathbf{h}_0 = -K_c^2 h_0, \tag{5.7}$$

and since α is independent of $\mathbf{x}$ (though dependent on t), this behaviour persists for all t, if only this single magnetic mode of maximum growth rate is considered. Hence (5.6) becomes

$$\frac{\partial \mathbf{h}_0}{\partial t} = \frac{u_0^2 l \alpha K_c}{\lambda} \mathbf{h}_0 - \lambda K_c^2 \mathbf{h}_0, \tag{5.8}$$

or, in terms of the magnetic energy density

$$M(t) = \tfrac{1}{2}\mathbf{h}_0^2, \tag{5.9}$$

$$\frac{dM}{dt} = \frac{(u_0^2 l)^2}{\lambda^3} \alpha_0(Q)\,[\alpha(Q,S) - \tfrac{1}{2}\alpha_0(Q)]\,M, \tag{5.10}$$

where now

$$S = 2tM(t)/\lambda.$$

For small t, $S \ll 1$, $\alpha \approx \alpha_0(Q)$, and M increases exponentially as described in §4. As t increases, S therefore increases, and so from (5.4), $\alpha(Q, S)$ decreases. However, α cannot decrease permanently below the level $\frac{1}{2}\alpha_0(Q)$, because if it did, equation (5.10) would imply an exponential decrease of M and so a decrease of S and so an immediate increase of α. Hence for $t \to \infty$, we must have

$$S \to S_0(Q) \tag{5.11}$$

where $S_0(Q)$ is determined by

$$\alpha(Q, S_0) = \tfrac{1}{2}\alpha_0(Q), \tag{5.12}$$

and is $O(1)$ for all Q (see appendix).

The maximum value of $M(t)$ attained before the ultimate decay

$$M(t) \sim \tfrac{1}{2}\lambda S_0(Q)\, t^{-1} \tag{5.13}$$

sets in, may now be estimated. For values of t such that $S(t) \ll 1$,

$$M(t) = M_0 \exp\{\tfrac{1}{2}\lambda^{-3}(\alpha_1(Q))^2 t\}, \quad \alpha_1 = u_0^2 l \alpha_0, \tag{5.14}$$

where M_0 is the initial energy density in the mode of maximum growth rate (assumed small). The functions (5.13) and (5.14) have the same order of magnitude

$$M_1 = \frac{1}{2\lambda^2} S_0(Q)\,(\alpha_1(Q))^2, \tag{5.15}$$

at a time of order

$$t_1 = \frac{\lambda^3}{(\alpha_1(Q))^2} \log \frac{S_0(Q)\,(\alpha_1(Q))^2}{2\lambda^2 M_0}, \tag{5.16}$$

and the maximum value attained by the magnetic energy density is therefore of order $M_1(Q)$. For $Q \ll 1$, the function (5.15) has the behaviour

$$M_1 \mathrel{\dot\sim} R_m^2 u_0^2, \tag{5.17†}$$

while for $Q \gg 1$, it has the behaviour

$$M_1 \mathrel{\dot\sim} Q^{-2} R \;\; u_0^2 = R_0^2 u_0^2. \tag{5.18}$$

† The symbol $\dot\sim$ is used to denote an asymptotic dependence with constants of order unity omitted.

In both limits, $M_1 \ll u_0^2$, so that the magnetic field does not in fact acquire more than a small fraction of the initially available kinetic energy of the motion. The rate of dissipation of kinetic energy via the Lorentz force to the ohmic sink is an order of magnitude greater than the rate of conversion of kinetic energy to magnetic energy. In this sense, the form of dynamo action considered is grossly inefficient, but even an inefficient dynamo is of course more significant than no dynamo at all.

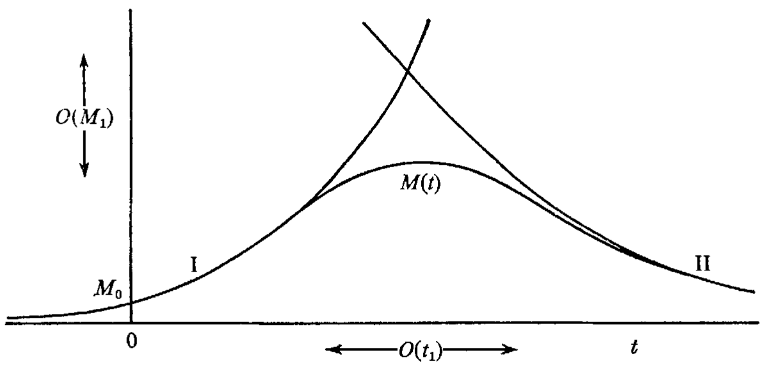

FIGURE 4. A sketch of the development of the magnetic energy density $M(t)$ in the mode of maximum initial growth rate. During state I, $t \ll t_1$, Lorentz forces are negligible; during stage II $t \gg t_1$, the magnetic energy decays because the velocity field which feeds it decays through ohmic dissipation.

We may now check on the validity of the condition (2.36), which will be satisfied for all $\mathbf{k}$ provided

$$|\mathbf{h}_0| \ll \lambda k \tag{5.19}$$

for all t, or equivalently provided

$$M_1 \ll \lambda^2 l^{-2} = R_m^{-2} u_0^2. \tag{5.20}$$

When $Q \ll 1$ (so that $R_m \ll 1$), this is certainly satisfied by virtue of (5.19). When $Q \gg 1$, from (5.18), it requires that

$$R_0^{-2} \gg R_m^2 = Q^2 R_0^2, \quad \text{i.e.} \quad R_0^{-4} \gg Q^2, \tag{5.21}$$

consistent with the requirement (4.19). It should be noted that when $Q \gg 1$, it is those inertial waves for which $\theta \approx \frac{1}{2}\pi$, $\cos\theta \approx 0$, which contribute most to the principal values of A_{ij}, so that the conditions (5.19) and (2.36) are virtually the same for those Fourier components of the velocity field which are of most crucial importance in the analysis.

6. Discussion

The analysis of the foregoing sections shows that a random superposition of inertial waves in a rotating fluid is certainly capable of transferring energy to an initially weak magnetic field, and it describes one mechanism by which this transfer may ultimately be limited by the intervention of Lorentz forces. The model however, suffers from the defect that it cannot predict the development of a steady state in which transfer of energy to the magnetic field is exactly balanced by ohmic dissipation. This is because no sources of energy are present in the model, and the presence of dissipation implies that ultimately the sum of kinetic and magnetic energies decreases to zero. In this sense, the model of this paper bears the same relation to the observed phenomenon of steady (or at least quasi-steady) geophysical and astrophysical dynamos as the theory of decaying homogeneous turbulence bears to the observed phenomenon of statistically steady shear flow turbulence: the results are suggestive and intrinsically interesting but are otherwise not of great value.

There are two ways in which the model may be modified so that a steady dynamo may result, but both modifications lead to major difficulties. The first, and simplest expedient, would be to introduce a random body force distribution $\mathbf{f}(\mathbf{x}, t)$ on the right-hand side of (2.1); but then in order to have a turbulent field of finite energy in the limit $\mathbf{h}_0 \to 0$, we have to include viscous dissipation also. A further difficulty is that the velocity field and so the properties of the growing magnetic field will be determined by the statistical properties of the assumed field $\mathbf{f}(\mathbf{x}, t)$; and unless some information is available concerning this, the labour involved in carrying out the calculation is hardly justified.

The second, and more realistic, way to supply energy to the fluid is to do so through the fluid boundaries, either by thermal or by mechanical means. (In the case of the fluid in the earth's core, both mechanisms are probably present. Thermal convection has long been considered an important mechanism in driving the irregular core motions that are inferred from, for example, secular variations of the surface magnetic field. Mechanical excitation can arise through relative motion of the core fluid and irregularities on the inner boundary of the mantle; and a recent analysis of the correlation between magnetic and gravitational perturbations on the surface of the earth (Hide & Malin 1970) suggests strongly that this also is an important mechanism.) A statistically steady state is then certainly conceivable, but unfortunately the idealization of spatial homogeneity must be abandoned, since the wave energy of the background turbulence must necessarily attenuate in the direction of energy propagation.

The mechanism of control of the growth of magnetic energy is (in this paper) very simple: where the growth is most rapid, the dissipation of the velocity field (which feeds the growing magnetic field) is likewise most rapid, and so the growth weakens. In the case of a steady dynamo, with a mechanical source of energy, as envisaged in the two preceding paragraphs, the control mechanism would be more subtle. The vital term $\nabla \wedge \overline{\mathbf{u} \wedge \mathbf{h}}$ in (1.9) arises essentially because $\mathbf{u}$ and $\mathbf{h}$ are out of phase; but as $\mathbf{h}_0$ grows in strength, the phase difference between $\mathbf{u}$ and $\mathbf{h}$ decreases (for a non-dissipative Alfvén wave, it vanishes altogether), and so $\nabla \wedge \overline{\mathbf{u} \wedge \mathbf{h}}$ will decrease until some kind of balance with the dissipative term $\lambda \nabla^2 \mathbf{h}_0$ of (1.9) is possible. The ultimate level of magnetic energy attained in these circumstances may well be very much larger than the maximum level M_1 attained under the conditions of §5 of this paper.

Appendix

We have to obtain the asymptotic behaviour for small and large Q of the function defined in (5.4), viz.

$$\alpha(Q,S) = \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi} \frac{\sin^3\theta\cos^2\phi}{1+4Q^2\cos^2\theta}\exp\left\{\frac{-2S\sin^2\theta\cos^2\phi}{1+4Q^2\cos^2\theta}\right\}d\theta\, d\phi. \quad (A\,1)$$

(i) $Q \ll 1$

In this limit, explicit dependence on Q disappears, and the integral may be most conveniently simplified by using polar angles θ', ϕ' measured from $\mathbf{h}_0$; the half-space $\theta < \frac{1}{2}\pi$ becomes the half-space $0 < \phi' < \pi$, and (A 1) becomes

$$\alpha(Q,S) \sim \alpha(O,S) = \frac{1}{4}\int_0^{\pi}\cos^2\theta'\, e^{-2S\cos^2\theta'}\sin\theta'\, d\theta'$$

$$= \frac{\pi^{\frac{1}{2}}}{16S}[(2S)^{-\frac{1}{2}}\operatorname{erf}(2S)^{\frac{1}{2}} - e^{-2S}]. \quad (A\,2)$$

(ii) $Q \gg 1$

For general Q, the ϕ-integration in (A 1) may be carried out in terms of the associated Bessel function $I_0(q)$ (Gradshteyn & Rijzhik 1965, §3.388):

$$\alpha(Q,S) = \frac{1}{8}\int_0^1 \frac{e^{-q}[3I_0(q) - I_0'(q)]}{1+4Q^2\mu^2}\,d\mu, \quad (A\,3)$$

where

$$q = \frac{S(1-\mu^2)}{1+4Q^2\mu^2}, \quad \mu = \cos\theta. \quad (A\,4)$$

For $Q \gg 1$, the dominant contribution comes from the neighbourhood of $\mu = 0$, and here

$$q \sim \frac{S}{1+4Q^2\mu^2}, \quad 2Q\mu \sim (S/q - 1)^{\frac{1}{2}}. \quad (A\,5)$$

Changing the variable of integration in (A 3) from μ to $x = q/S$, we have, for $Q \to \infty$,

$$\alpha \sim g(S)/32Q, \quad (A\,6)$$

where

$$g(S) = \int_0^1 \frac{e^{-Sx}[3I_0(Sx) - I_0'(Sx)]}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\,dx. \quad (A\,7)$$

The functions defined by (A 2) and (A 7) are monotonic decreasing functions of S (as is clearly the general expression (A 1)). The function $S_0(Q)$ defined by (5.12) is clearly $O(1)$ as $Q \to 0$ and as $Q \to \infty$; and since $\alpha(Q,S)$ decreases more rapidly with S when Q is large than when Q is small (the integral (A 1) being then dominated by contributions from the neighbourhood of $\theta = \frac{1}{2}\pi$), $S_0(Q)$ is also monotonic decreasing, and $O(1)$ for all Q.

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